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Quantization and contact structure on manifolds with projective structure

Indranil Biswas [a,*], Rukmini Dey [b]

[a] School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
[b] Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India

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Abstract

We consider complex manifolds with a class of holomorphic coordinate functions satisfying the condition that each transition function is given by the standard action on $\mathbb{CP}^{2n-1}$ of some element in $\mathrm{Sp}(2n, \mathbb{C})/\mathbb{Z}_2$. We show that such a manifold has a natural contact structure. Given any contact manifold, one can associate with it a symplectic manifold. It is shown that the symplectic manifolds arising from complex manifolds with special coordinate functions of the above type admit a canonical quantization. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let V be complex vector space of dimension $2n+2$ and $\omega \in \wedge^2 V^*$ a symplectic form on V. The space of all automorphisms of V preserving ω will be denoted by $\mathrm{Sp}(V)$. The center of $\mathrm{Sp}(V)$ is $\mathbb{Z}_2$ consisting of $\pm\mathrm{Id}_V$. The quotient $\mathrm{Sp}(V)/\mathbb{Z}_2$ will be denoted by G. It acts faithfully on the projective space $P(V)$ of lines in V.

Let M be a complex manifold of dimension $2n+1$. A G-structure on M is a covering of M by holomorphic coordinate charts $\{U_i, \phi_i\}_{i\in I}$, where $\phi_i : U \to P(V)$ is a biholomorphism with the image, such that each $\phi_i \circ \phi_j^{-1}$ is the restriction of the action of some $T_{i,j} \in G$ on $P(V)$. If M is equipped with a G-structure $\mathfrak{p}$, then we construct a contact structure $F(\mathfrak{p}) \subset \mathrm{TM}$ on M.

* Corresponding author.
E-mail addresses: indranil@math.tifr.res.in (I. Biswas), rkmn@mri.ernet.in (R. Dey).

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Let $F \subset \mathrm{TZ}$ be a general contact structure on a complex manifold Z, not necessarily of the above type. Denoting the quotient TZ/F by N, let $N' := N \setminus \{0\}$ be the complement of the zero section. The space N' has a natural symplectic form arising from the contact structure.

Let $N'(\mathfrak{p})$ denote the symplectic manifold corresponding to the contact structure $F(\mathfrak{p})$ defined above. We prove that $N'(\mathfrak{p})$ admits a canonical quantization (Theorem 4.2).

In [15], quantization for a Riemann surface with projective structure was considered.

2. Contact structure and projective space

2.1. Contact structure

Let M be a complex manifold of odd dimension, say $2n + 1$. Its holomorphic tangent bundle will be denoted by TM. A contact structure on M is a holomorphic subbundle $F \subset \mathrm{TM}$ of rank $2n$ which is maximally nonintegrable. To explain this nonintegrability condition, let N denote the normal bundle TM/F, and let

$$q\colon \mathrm{TM} \to N \tag{2.1}$$

be the obvious quotient map. We have a homomorphism

$$\psi\colon F \otimes F \to N$$

that sends $s_1 \otimes s_2$ to $q([s_1, s_2])$, where s_1 and s_2 are any pair of (local) sections of F and $[s_1, s_2]$ is the Lie bracket. It is easy to see that the two identities for Lie bracket

1. $[s_1, s_2] = -[s_2, s_1]$,
2. $[fs_1, s_2] = f[s_1, s_2] - \langle \mathrm{d}f, s_2\rangle s_1$

ensure that ψ is a homomorphism of vector bundles. The point to note is that $q(\langle \mathrm{d}f, s_2\rangle s_1) = 0$.

The subbundle F is called *maximally nonintegrable* if the bilinear form on F defined by ψ is nondegenerate. Since ψ is antisymmetric, the nondegeneracy condition implies that the dimension of F must be even.

Let ω be a nowhere vanishing one-form on an open subset U of M such that for every $x \in U$, the kernel of the homomorphism

$$\omega(x)\colon T_xM \to \mathbb{C}$$

coincides with the subspace $F_x \subset T_xM$. Note that fixing such a form is equivalent to fixing a trivialization of the line bundle N over U. The evaluation of ω on N defines the corresponding trivialization of N. Conversely, if we trivialize N over U, then the quotient homomorphism q in (2.1) becomes a one-form on U.

Consider the top form $\bar{\omega} := \omega \wedge (\mathrm{d}\omega)^n$ on U. It can be shown that the condition that $\bar{\omega}$ is nowhere vanishing on U depends only on F and is independent of the choice of ω. Indeed, if we substitute ω by $\theta = f\omega$, where f is a smooth function on U, then $\mathrm{d}\theta = f\,\mathrm{d}\omega + \mathrm{d}f \wedge \omega$. Using this and the fact $\omega \wedge \omega = 0$ we conclude that $\theta \wedge (\mathrm{d}\theta)^n = f^{n+1}\bar{\omega}$.

It is easy to check that F is maximally nonintegrable if and only if M can be covered by open subsets with trivializations of N over them such that on each of the open sets the corresponding form $\bar{\omega}$ is nowhere vanishing.

The projection of the total space of the normal bundle N to M will be denoted by p. Let

$$N' := N \setminus \{0\}$$

be the complement of the zero section of the total space of N. The restriction of p to N' will also be denoted by p. The pullback line bundle p^*N over N' is evidently trivial. Indeed, p^*N has a tautological section over N which does not vanish anywhere on N'.

Using the trivialization of p^*N over N', the projection q in (2.1) defines a one-form on N'. To define the one-form in details, for any $z \in N'$, let $\mathrm{d}p(z){:}T_zN' \to T_{p(z)}M$ be the differential of the projection p. Since $p^{-1}(p(z)) = \mathbb{C}z$, the vector z identifies the fiber $N_{p(z)}$ with $\mathbb{C}$. Therefore, the composition homomorphism

$$q \circ \mathrm{d}p(z) : T_zN' \to N_{p(z)} = \mathbb{C}$$

defines a one-form on N'. Let θ denote this holomorphic one-form on N'. Clearly θ is nowhere vanishing.

Proposition 2.1. *A subbundle F of TM of corank one is a contact structure if and only if the two-form $\mathrm{d}\theta$ on N' is a symplectic form.*

Proof. Take a sufficiently small open subset U of M. Fix a section $s : U \to N'$. So $p \circ s$ is the identity map of U. Let f denote the function on $p^{-1}(U) \subset N'$ that sends any z to the complex number c with the property $cz = s(p(z))$.

Since the section s trivializes the line bundle N over U, the quotient homomorphism q defines a one-form on U. Let ω denote this one-form on U. It is straight-forward to check that the identity

$$\theta = fp^*\omega$$

is valid. Consequently, we have

$$\mathrm{d}\theta = fp^*\,\mathrm{d}\omega + \mathrm{d}f \wedge p^*\omega. \tag{2.2}$$

Now, from (2.2) we immediately have the identity

$$(\mathrm{d}\theta)^{n+1} = f^n\,\mathrm{d}f \wedge p^*(\omega \wedge (\mathrm{d}\omega)^n) \tag{2.3}$$

of top forms on $p^{-1}(U) \subset N'$. If $\mathrm{d}\theta$ is a symplectic form then $(\mathrm{d}\theta)^{n+1}$ is nowhere vanishing. In that case (2.3) implies that $\omega \wedge (\mathrm{d}\omega)^n$ is nowhere vanishing. In other words, F is a contact structure.

Conversely, if F is a contact structure, then first observe that if $(x_1, x_2, \ldots, x_{2n+1})$ is a holomorphic coordinate function on U, then $(f, x_1 \circ p, x_2 \circ p, \ldots, x_{2n+1} \circ p)$ is a holomorphic coordinate function on $p^{-1}(U)$. Consequently, from (2.3) it follows immediately that if $\omega \wedge (\mathrm{d}\omega)^n$ is nowhere vanishing, then $(\mathrm{d}\theta)^{n+1}$ is also nowhere vanishing on $p^{-1}(U)$. Since $\mathrm{d}\theta$ is closed, this implies that if F is a contact structure then $\mathrm{d}\theta$ is a symplectic form. This completes the proof of the proposition. □

We will give an alternative description of the form θ.

On the total space Ω^1_M of the holomorphic cotangent bundle there is a canonical one-form which is defined as follows. Denoting the projection of Ω^1_M to M by f, consider the differential $\mathrm{d}f(z): T_z\Omega^1_M \to T_{f(z)}M$ of f at a point $z \in \Omega^1_M$. The composition

$$T_z\Omega^1_M \xrightarrow{\mathrm{d}f(z)} T_{f(z)}M \xrightarrow{z} \mathbb{C}$$

defines a one-form on Ω^1_M which will be denoted by η.

Consider the dual homomorphism $q^{\star}: N^* \to \Omega^1_M$ of the homomorphism q in (2.1). The complement $N^* \setminus \{0\}$ of the zero section is identified with N' defined earlier. Indeed, any $z \in N' \cap p^{-1}(x)$ identifies the fiber N_x, hence its dual N^*_x, with $\mathbb{C}$. Let

$$g: N' \to N^* \setminus \{0\} \tag{2.4}$$

be the isomorphism that sends any z to the element in $N^*_{p(z)}$ corresponding to 1 for the trivialization of it defined by z.

The following lemma is obvious after unraveling the definitions.

Lemma 2.2. *The one-form* θ on N' coincides with $(q^{\star} \circ g)^*\eta$.

The Lemma 2.2 gives the following reformulation of Proposition 2.1: a subbundle F of TM of corank one is a contact structure if and only if the two-form $(q^{\star} \circ g)^*\,\mathrm{d}\eta$ on N' is a symplectic form.

2.2. Contact structure on projective space

Let V be a complex vector space of dimension $2n+2$ equipped with a symplectic form ω. In other words, ω is an anti-symmetric nondegenerate bilinear form on V.

Let $P(V)$ denote the projective space consisting of all one-dimensional subspaces of V. The natural projection of $V \setminus \{0\}$ to $P(V)$ will be denoted by π.

Using the symplectic form ω, we will construct a contact structure on $P(V)$.

Take any line $\alpha \in P(V)$ in V. Consider the hyperplane

$$V_\alpha = \alpha^\perp := \{v \in V \mid \omega(v, \alpha) = 0\}.$$

Since ω is antisymmetric, α is contained in V_α. Therefore, the image of V_α by the differential $\mathrm{d}\pi$ of the projection π is a hyperplane in the tangent space $T_\alpha P(V)$. Note that this hyperplane of $T_\alpha P(V)$, which we will henceforth denote by F_α, does not depend on the choice of the vector in the line α at which the differential $\mathrm{d}\pi$ is considered.

Let F denote the holomorphic subbundle of the tangent bundle $TP(V)$ whose fiber over any $\alpha \in P(V)$ is the hyperplane F_α constructed above.

For any $v \in V$, the holomorphic tangent space T_vV is identified with V. Hence on $V \setminus \{0\}$ we have the tautological vector field that associates w to any $w \in V \setminus \{0\}$. This vector field will be denoted by τ. Let

$$\gamma := i_\tau\omega \tag{2.5}$$

be the one-form on $V \setminus \{0\}$ obtained by contracting ω by the vector field τ.

Consider the space N' equipped with one-form θ constructed in Section 2.1 from the subbundle F.

Proposition 2.3. *There is a natural degree two covering map β from $V \setminus \{0\}$ to N'. If $\beta(v) = \beta(w)$, then $v = \pm w$. The map β has the property that $p \circ \beta = \pi$, where p as before is the obvious projection of N' to $P(V)$. Furthermore, $\beta^*\theta$ coincides with the one-form γ defined in (2.5).*

Proof. Take any vector $v \in V \setminus \{0\}$. Let $\alpha := \mathbb{C}v$ be the line in V defined by v.

Recall the projection $\pi: V \setminus \{0\} \to P(V)$. Consider its differential

$$\mathrm{d}\pi(v): V \to T_\alpha P(V)$$

at the point v. Take any $w \in V$ such that

$$\omega(v, w) \neq 0.$$

Now send v to

$$\frac{1}{\omega(v, w)} q(\mathrm{d}\pi(v)(w)) \in N_\alpha,$$

where $N_\alpha = T_\alpha P(V)/F_\alpha$ and q, as in (2.1), is the projection of $T_\alpha P(V)$ to N. First observe that since the kernel of $\mathrm{d}\pi(v)$ is the line α and $\omega(v, v) = 0$, we have $\mathrm{d}\pi(v)(w) \neq 0$. Furthermore, since the subspace $F_\alpha \subset T_\alpha P(V)$ is the image of V_α and $w\neg \in V_\alpha$, we have

$$q(\mathrm{d}\pi(v)(w)) \neq 0.$$

So, $q(\mathrm{d}\pi(v)(w))/\omega(v, w)$ is a nonzero element in the fiber N_α.

If we substitute w by $w' = cw$, where c is a complex number, then clearly $q(\mathrm{d}\pi(v)(w'))/\omega(v, w')$ coincides with $q(\mathrm{d}\pi(v)(w))/\omega(v, w)$. If we substitute w by any w' satisfying the condition $\omega(v, w') \neq 0$, then there is a nonzero complex number c such that $cw - w' \in V_\alpha$. This implies that $q(\mathrm{d}\pi(v)(w')) = cq(\mathrm{d}\pi(v)(w))$. Therefore, the vector $q(\mathrm{d}\pi(v)(w))/\omega(v, w)$ is independent of the choice of w.

Let

$$\beta: V \setminus \{0\} \to N'$$

be the map that sends any v to $q(\mathrm{d}\pi(v)(w))/\omega(v, w)$. Since the differential $\mathrm{d}\pi(-v)$ coincides with $-\mathrm{d}\pi(v)$, it follows immediately that $\beta(v) = \beta(-v)$. It is easy to check that if $\beta(v) = \beta(v')$, then either $v = v'$ or $v = -v'$.

This map β is the degree two covering map asserted in the proposition. Clearly, $p \circ \beta = \pi$. To complete the proof we need to show that $\beta^*\theta = \gamma$. It may be noted that if the symplectic form ω is replaced by $c\omega$, where $c \in \mathbb{C} \setminus \{0\}$, then the subbundle F remains unchanged, but the map β changes by multiplication with $1/c$.

For any line α in V and vector $v \in \alpha$, let

$$f_v: \frac{V}{V_\alpha} \to \mathbb{C}$$

be the functional defined by $v' \mapsto \omega(v, v')$. Since $N_\alpha = V/V_\alpha$, we have a map

$$f: V \setminus \{0\} \to N^*$$

that sends any v to the functional f_v.

It is easy to see that f coincides with $g \circ \beta$, where the map g is defined in (2.4). Indeed, this follows immediately from the fact that in the identification $N_\alpha = V/V_\alpha$, the vector $cv'/f_v(v')$, where $c \in \mathbb{C}$ and v' a nonzero vector in V/V_α, corresponds to $c\beta(v)$.

Now in view of Lemma 2.2 it suffices to prove that $(q^\star \circ f)^*\eta$ coincides with γ, where $q^\star$ is defined in Section 2.1. But the identity $(q^\star \circ f)^*\eta = \gamma$ is immediate after unraveling the definitions. This completes the proof of the proposition. □

The following proposition is deduced from Propositions 2.1 and 2.3.

Proposition 2.4. *The subbundle F of $TP(V)$ defines a contact structure on $P(V)$.*

Proof. In view of Proposition 2.1 and the assertion in Proposition 2.3 that $\beta^*\theta = \gamma$, it suffices to prove that $\mathrm{d}\gamma$ is a symplectic form on V'. In fact, we will show that

$$\mathrm{d}\gamma = 2\omega. \tag{2.6}$$

Take any $u, v \in V$ and denote the corresponding constant vector fields on V also by u and v. To prove (2.6), first observe that

$$\mathrm{d}\gamma(u, v) = (\mathrm{d}i_\tau\omega)(u, v) = (L_\tau\omega)(u, v) = -\omega(L_\tau u, v) - \omega(u, L_\tau v),$$

where L denotes the Lie derivative. But $L_\tau u = [\tau, u] = -u$. Therefore, (2.6) is proved. This completes the proof of the proposition. □

Note that if the symplectic form ω is replaced by its nonzero scalar multiple, then the contact structure remains unchanged.

The form γ has the following description in terms of local coordinates.

Let z_i, $i = 1, \dots, 2n+2$, be a complex basis of linear functionals on V such that

$$\omega = \sum_{1 \le i < j \le 2n+2} \mathrm{d}z_i \wedge \mathrm{d}z_j.$$

Let $\alpha = \mathbb{C}(x_1, \dots, x_{2n+2})$ be a line in V expressed in terms of the dual basis of V. We have $V_\alpha = \{(z_1, \dots, z_{2n+2}) \in V \mid \sum_{i<j}(x_i z_j - x_j z_i) = 0\}$. From this it can be shown that $\gamma = \sum_{i<j}(z_j\,\mathrm{d}z_i - z_i\,\mathrm{d}z_j)$.

3. Projective structure and contact structure

As in Section 1, let V be a $2n+2$ dimensional complex vector space equipped with a symplectic form ω. Let $\mathrm{Sp}(V)$ denote the group of all automorphisms of the vector space V preserving the symplectic form ω. The center of $\mathrm{Sp}(V)$ is $\mathbb{Z}_2 = \pm\mathrm{Id}_V$. The quotient $\mathrm{Sp}(V)/\mathbb{Z}_2$ will be denoted by G. The group G acts on $P(V)$ as automorphisms. This action is faithful.

Let M be a complex manifold of dimension $2n+1$. A G-structure on M is defined by giving a covering of M by holomorphic charts, say $\{U_i, \phi_i\}_{i\in I}$, where ϕ_i is a biholomorphism from the open subset U_i of M to an open subset of $P(V)$, such that for every pair

$i, j \in I$, there is $T_{j,i} \in G$ with the property that the composition $\phi_j \circ \phi_i^{-1}$ is the restriction of the automorphism $T_{j,i}$ of $P(V)$ to $\phi_i(U_i \cap U_j)$. Two such G-structures $\{U_i, \phi_i\}_{i \in I}$ and $\{U_i, \phi_i\}_{i \in I'}$ are called *equivalent* if their union $\{U_i, \phi_i\}_{i \in I \cup I'}$ is again a G-structure. By a *G-structure* on M we will always mean an equivalence class in the above sense. Therefore, given a G-structure $\mathfrak{p}$ on M, there is a maximal atlas $\{U_i, \phi_i\}_{i \in I}$ with the above property of transition functions. Any coordinate function (U, ϕ) will be called *compatible* with $\mathfrak{p}$ if it is in the maximal atlas.

If $n = 1$, then $G = \mathrm{PSL}(2, \mathbb{C})$. Therefore, a G-structure on a Riemann surface is a projective structure in the usual sense [8]. See [11, Chapter 8] for more general G-structures.

Let M be equipped with a G-structure which we will denote by $\mathfrak{p}$.

The action of G on $P(V)$ preserves the contact structure F on $P(V)$ obtained in Proposition 2.4. Therefore, the G-structure $\mathfrak{p}$ induces a contact structure on M. This contact structure will be denoted by $F(\mathfrak{p})$. So for any coordinate chart (U_i, ϕ_i) for $\mathfrak{p}$, the restriction of the subbundle

$$F(\mathfrak{p}) \subset \mathrm{TM}$$

to U_i is simply the inverse image of the subbundle $F|_{\phi_i(U_i)} \subset T\phi_i(U_i)$ by the differential $\mathrm{d}\phi_i$.

Let $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ denote the symplectic manifold corresponding to $F(\mathfrak{p})$ obtained in Proposition 2.1. In the next section we will describe a canonical quantization of this symplectic structure.

Using Proposition 2.3 we will give another description of $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$.

Let V' denote the quotient of $V \setminus \{0\}$ obtained by identifying any vector v with $-v$. Let $\pi' : V' \to P(V)$ denote the obvious projection. Since the form ω on V is invariant under the automorphism $-\mathrm{Id}_V$, it descends as a symplectic form on V'. This descended form on V' will be denoted by ω'.

Using the map β in Proposition 2.3, the space V' gets identified with N'. Furthermore, this identification takes the projection p to π'. In (2.6) we saw that $\mathrm{d}\gamma = 2\omega$. Therefore, the identification of N' with V' using β takes the form $\mathrm{d}\theta$ on N' to $2\omega'$.

The obvious action of $\mathrm{Sp}(V)$ on V induces an action of G on V' which preserves the form ω'. The projection π' is equivariant for the actions of G on V' and $P(V)$. Since the contact structure F on $P(V)$ obtained in Proposition 2.4 is invariant under the action of G on $P(V)$, we conclude that the action of G on $P(V)$ lifts to an action on N'. It is immediate that the action of G on N' preserves the one-form θ. The identification of V' with N' is evidently G-equivariant.

If $\{U_i, \phi_i\}_{i \in I}$ is a covering of M by coordinate charts compatible with the G-structure $\mathfrak{p}$, then for each $i \in I$, consider the open subset $\pi'^{-1}(\phi_i(U_i)) \subset V'$. For any ordered pair $i, j \in I$, the action of $T_{j,i} := \phi_j \circ \phi_i^{-1} \in G$ on V' identifies $\pi'^{-1}(\phi_i(U_i \cap U_j))$ with $\pi'^{-1}(\phi_j(U_i \cap U_j))$. Therefore, we may glue $\pi'^{-1}(\phi_i(U_i))$ and $\pi'^{-1}(\phi_j(U_j))$ using the isomorphism $T_{i,j}$ of $\pi'^{-1}(\phi_i(U_i \cap U_j))$ with $\pi'^{-1}(\phi_j(U_i \cap U_j))$.

Since $T_{i,j} = T_{j,i}^{-1}$ and $T_{i,j}T_{j,k}T_{k,i} = e$, the combination of all these gluing produce a symplectic manifold W equipped with a symplectic form Θ. The symplectic form is constructed from ω' using its G-invariance property. Furthermore, the projection π' being G-equivariant induces a projection ψ of W to M. From Proposition 2.3 it follows immediately that W is identified with $N'(\mathfrak{p})$ which takes the symplectic form $\mathrm{d}\theta(\mathfrak{p})$ to Θ. Furthermore, this identification of W with $N'(\mathfrak{p})$ takes the projection ψ to the obvious projection of $N'(\mathfrak{p})$ to M.

4. Quantization of symplectic structure

4.1. Definition of quantization

Let Z be a complex manifold equipped with a holomorphic symplectic form Ω. Let $C(Z)$ denote the commutative algebra consisting of all complex valued smooth functions on Z. The symplectic form Ω defines a Poisson structure on $C(Z)$ which is defined as follows.

Since the bilinear pairing Ω on TZ is nondegenerate, it defines a bilinear pairing on the cotangent bundle Ω^1_Z which will be denoted by Ω^{-1}. The Poisson structure is defined by sending any pair of functions f and g in $C(Z)$ to

$$\{f, g\} := \Omega^{-1}(\mathrm{d}f, \mathrm{d}g). \tag{4.1}$$

This Poisson structure makes $C(Z)$ into a Lie algebra satisfying the Leibniz identity $\{fg, k\} = f\{g, k\} + g\{f, k\}$.

Let $\mathcal{A}(Z) := C(Z)[[h]]$ be the space of all formal Taylor series

$$f := \sum_{j=0}^{\infty} h^j f_j,$$

where $f_j \in C(Z)$ and h is a formal variable.

A *quantization* of the symplectic form Ω is an associative algebra operation on $\mathcal{A}(Z)$ satisfying certain conditions [1,7,12]. For another element $g := \sum_{j=0}^{\infty} h^j g_j \in \mathcal{A}(Z)$, if

$$f \star g = \sum_{j=0}^{\infty} h^j c_j$$

is the multiplication, then the conditions in question say:

1. each c_i is some polynomial (independent of f and g) in derivatives (of arbitrary order) of $\{f_j\}_{j\geq 0}$ and $\{g_j\}_{j\geq 0}$;
2. $c_0 = f_0 g_0$;
3. $1 \star f = f \star 1 = f$ for every $f \in C(Z)$;
4. $f \star g - g \star f = \sqrt{-1}h\{f_0, g_0\} + h^2 k$, where $k \in \mathcal{A}(Z)$ depends on f, g.

The first condition implies that the quantization is local in the sense that the restriction of $f \star g$ to an open subset U of Z depends only on $f|_U$ and $g|_U$. In other words, if $f|_U = f_1|_U$ and $g|_U = g_1|_U$, then $(f \star g)|_U = (f_1 \star g_1)|_U$. The second condition says that the algebra is a deformation, parametrized by the variable h, of the usual commutative algebra structure of $C(Z)$. The third condition says that the derivative, with respect to the variable h, of the algebra operation coincides with the Poisson bracket.

It is known that all symplectic manifolds admit quantizations [4,5,7]. However, there is no uniqueness of quantization. In fact, if $\dim Z \geq 2$, then there are infinitely many distinct quantizations of Ω.

We will now describe a very well-known quantization called the Moyal–Weyl quantization.

4.2. *The Moyal–Weyl quantization*

Consider the symplectic vector space (V, ω). We will think of V as a symplectic manifold with symplectic structure ω. As in Section 4.1, let $C(V)$ denote the space of smooth complex valued functions on V equipped with the Poisson structure defined in (4.1).

Let

$$\Delta : V \to V \times V$$

denotes the diagonal homomorphism defined by $v \mapsto (v, v)$. There exists a unique differential operator

$$D : C(V \times V) \to C(V \times V) \tag{4.2}$$

with constant coefficients such that for $f, g \in C(V)$,

$$\{f, g\} = \Delta^* D(f \otimes g),$$

where $f \otimes g$ is the function on $V \times V$ defined by $(u, v) \mapsto f(u)g(v)$.

The *Moyal–Weyl algebra* is defined by

$$f \star g = \Delta^* \exp(\tfrac{1}{2}\sqrt{-1}hD)(f \otimes g) \in \mathcal{A}(V) \tag{4.3}$$

for $f, g \in C(V)$, and it is extended to a multiplication operation on $\mathcal{A}(V)$ using bilinearity condition with respect to h. In other words, if $f := \sum_{j=0}^{\infty} h^j f_j$ and $g := \sum_{j=0}^{\infty} h^j g_j$ are two elements of $\mathcal{A}(V)$, then

$$f \star g = \sum_{i,j} h^{i+j} (f_i \star g_j) \in \mathcal{A}(V).$$

It is known that this operation $\star$ defined above makes $\mathcal{A}(V)$ into an associative algebra that quantizes the symplectic structure ω. See [12] for the details.

The Poisson structure on $C(V)$ and the differential operator D has the following expression in terms of a symplectic basis $\{z_i\}_{1\leq i\leq 2n+2}$ of functionals on V.

If $\omega = \frac{1}{2}\sum_{i,j} \omega_{ij}\, \mathrm{d}z_i \wedge \mathrm{d}z_j$, then

$$\{f, g\} = \sum_{i,j} t_{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j},$$

where (t_{ij}) is the inverse matrix of (ω_{ij}). The operator $\Delta^* D$ has the expression

$$\Delta^* D(f \otimes g)(z) = \sum_{i,j} t_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} (f(x)g(y))|_{y=x=z}$$

(see [12]), where x_i (respectively, y_i) are the linear coordinates on the first (respectively, second) factor of $V \times V$ obtained from z_i. For $f, g \in C(V)$, the Moyal–Weyl product $f \star g$ has the expression

$$(f \star g)(z) = \sum_k \left(\frac{1}{k!} \left(\frac{\sqrt{-1}}{2} \sum_{i,j} t_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} \right)^k (f(x)g(y))|_{y=x=z} \right) h^k.$$

We will use the Moyal–Weyl quantization in quantizing the symplectic manifold $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ associated to a manifold M with a G-structure.

4.3. *Quantization on projective space*

Consider the contact structure F on $P(V)$ constructed in Proposition 2.4. From Proposition 2.1 we know that $(N', \mathrm{d}\theta)$ in Proposition 2.3 is a symplectic manifold.

In Section 3 we saw that N' is identified with the quotient $V' = (V \setminus \{0\})/\mathbb{Z}_2$ taking $\mathrm{d}\theta$ to the symplectic form $2\omega'$ on V' defined in Section 3.

The involution $-\mathrm{Id}_V$ of V preserves the symplectic form 2ω and hence the differential operator D in (4.2) for 2ω satisfies the identity

$$D(f)(-v, -w) = D(-f)(v, w) \tag{4.4}$$

for every function $f \in C(V \times V)$.

Consider the Moyal–Weyl quantization of $C(V)$, defined in Section 4.2, for the symplectic form 2ω on the vector space V. The identity (4.4) ensures that the Moyal–Weyl quantization for 2ω is invariant under the involution $-\mathrm{Id}_V$ of V. Consequently, it descends to a quantization of the symplectic form $2\omega'$ on V'.

Now using the above mentioned identification of $(V', 2\omega')$ with $(N', \mathrm{d}\theta)$, the quantization of $2\omega'$ on V' gives a quantization of the symplectic form $\mathrm{d}\theta$ on N'. Let

$$\star : \mathcal{A}(N') \underset{\mathbb{C}}{\otimes} \mathcal{A}(N') \to \mathcal{A}(N') \tag{4.5}$$

be the above obtained quantization of the symplectic structure $\mathrm{d}\theta$.

Since 2ω is invariant under the action of $\mathrm{Sp}(V)$ on V, from the definition of D for 2ω given in (4.2) if follows that for any $\tau \in \mathrm{Sp}(V)$,

$$D(T_\tau \circ f) = T_\tau \circ D(f),$$

where T_τ denotes the diagonal action of τ on $V \times V$ and $f \in C(V \times V)$. This identity implies that the Moyal–Weyl product for 2ω defined in (4.3) satisfies the identity

$$(f \circ \tau) \star (g \circ \tau) = (f \star g) \circ \tau$$

for every $f, g \in C(V)$ and every $\tau \in \mathrm{Sp}(V)$.

We noted in Section 3 that the action of G on N' is the transport of the action of G on V'. Therefore, we have the following proposition.

Proposition 4.1. *The quantization of the symplectic manifold* $(N', \mathrm{d}\theta)$ *constructed in* (4.5) *has the property that for every* $\tau \in G$ *and* $f, g \in \mathcal{A}(N')$, *the identity*

$$(f \circ \tau)\star(g \circ \tau) = (f \star g) \circ \tau$$

is valid.

This G invariance property of $\star$ enables us to construct a quantization for any G-structure.

4.4. *G-structure and quantization*

As in Section 3, let M be a complex manifold equipped with a G-structure $\mathfrak{p}$. Let $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ be the symplectic manifold constructed in Section 3 from $\mathfrak{p}$. The obvious projection of $N'(\mathfrak{p})$ to M will be denoted by γ. As in Section 3, the contact structure on M corresponding to $\mathfrak{p}$ will be denoted by $F(\mathfrak{p})$.

Take a holomorphic coordinate chart $\phi{:}U \to P(V)$ of M compatible with the given G-structure $\mathfrak{p}$. Therefore, ϕ takes the contact structure $F(\mathfrak{p})|_U$ to $F|_{\gamma(U)}$, where F is the contact structure on $P(V)$. Therefore, $\gamma^{-1}(U)$ gets identified with $p^{-1}(\gamma(U))$, where p is the obvious projection of N' to $P(V)$. This identification, which we will denote by ψ, evidently takes the symplectic form $\mathrm{d}\theta$ on $p^{-1}(\gamma(U))$ to the symplectic form $\mathrm{d}\theta(\mathfrak{p})$ on $\gamma^{-1}(U)$. Consequently, using ψ, the quantization of $\mathrm{d}\theta$ constructed in (4.5) gives a quantization of the symplectic form $\mathrm{d}\theta(\mathfrak{p})$ over U.

If we have another holomorphic coordinate chart $\phi'{:}U \to P(V)$ compatible with $\mathfrak{p}$, then $\phi' \circ \phi^{-1}$ coincides with the action of some $g \in G$ on $P(V)$. Now Proposition 4.1 implies that the new quantization of $\mathrm{d}\theta(\mathfrak{p})$ over $\gamma^{-1}(U)$ corresponding to ϕ' actually coincides with the previous one constructed from ϕ. Recall that the Moyal–Weyl quantization is local, as asserted by the first condition on $\star$ in Section 4.1.

Therefore, if we cover M by holomorphic coordinate charts $\{U_i, \phi_i\}$ compatible with $\mathfrak{p}$, then the quantization of $\mathrm{d}\theta(\mathfrak{p})$ over individual open sets $\gamma^{-1}(U_i)$ constructed using the corresponding coordinate function ϕ_i patch compatibly. In other words, on each $\gamma^{-1}(U_i \cap U_j)$, the quantizations constructed using ϕ_i and ϕ_j coincide.

Thus we have proved the following theorem.

Theorem 4.2. *For any G-structure* $\mathfrak{p}$ *on M, the corresponding symplectic manifold* $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ *admits a canonical quantization.*

A given complex manifold may admit more than one G-structure. For example, when M is a Riemann surface, the space of all projective structures on M is an affine space for $H^0(M, K_M^{\otimes 2})$, the space of quadratic differentials on M [8]. We remarked earlier that a G-structure on a Riemann surface is same as a projective structure. Therefore, a connected Riemann surface admits exactly one projective structure if and only if it is compact with genus 0. Using the uniformization theorem it is easy to show that any Riemann surface admits a projective structure.

If M is a Riemann surface with a G-structure $\mathfrak{p}$, then $N'(\mathfrak{p})$ is the space of nonzero tangent vectors of M. Therefore, $N'(\mathfrak{p})$ does not depend on $\mathfrak{p}$. In fact, even the symplectic structure $\mathrm{d}\theta(\mathfrak{p})$ on $N'(\mathfrak{p})$ is independent of $\mathfrak{p}$. The space of nonzero tangent vectors of a Riemann surface can be identified with the space of nonzero cotangent vectors. But the cotangent bundle has a canonical symplectic structure. The symplectic form $\mathrm{d}\theta(\mathfrak{p})$ coincides with this canonical symplectic form. However, it can be proved that the quantization of $\mathrm{d}\theta(\mathfrak{p})$ constructed in Theorem 4.2 determines the projective structure $\mathfrak{p}$.

To each Riemann surface with projective structure, there are some associated differential operators D_i of order i, where $i \geq 2$ [2, Theorem 4.1]. The second-order operator D_2 is the Liouville operator that determines the projective structure. The quantization in Theorem 4.2 determines these operators D_i. Consequently, given a quantization corresponding to a projective structure, the quantization determines the projective structure uniquely. We recall that the space of all equivalence classes of projective structures on a Riemann surface M is an affine space for the space of all quadratic differential on M [8]. In particular, if M is a compact Riemann surface of genus g, then the space of all equivalence classes of projective structures on M is a complex affine space of dimension $3g-3$.

To construct examples of G structures on higher dimensional complex manifolds, consider the group $\mathrm{SU}(1, 2n+1)$ that preserves the sesquilinear form $-|z_1|^2 + \sum_{i=2}^{2n+2} |z_i|^2$ on $\mathbb{C}^{2n+2}$. Its action on $\mathbb{CP}^{2n+1}$ preserves the unit ball in $\mathbb{C}^{2n+1} \subset \mathbb{CP}^{2n+1}$. We recall that a theorem of Yau asserts that a smooth projective manifold X of dimension $2n+1$ is isomorphic to the quotient of the unit ball by a torsionfree discrete subgroup of $\mathrm{SU}(1, 2n+1)$ if and only if the canonical bundle K_X is ample and $(2n+1)c_1(X)^{2n+1} = 4(n+1)c_1(X)^{2n-1}c_2(X)$ [14]. Consider the intersection

$$G' := \mathrm{SU}(1, 2n+1) \cap \mathrm{Sp}(2n+2, \mathbb{C}).$$

Now if Γ is a torsionfree discrete subgroup of G' then quotient of the unit ball by the action of Γ has a natural G-structure. In fact instead of G' we can also take the intersection of $\mathrm{SU}(1, 2n+1)$ with the group preserving any given symplectic form on $\mathbb{C}^{2n+2}$ not necessarily the standard one.

However, it should clarified that although in the case dimension one any Riemann surface admits a projective structure, in higher dimensions, the condition of existence of a G-structure imposes restrictions on the Chern classes on the underlying complex manifold (see e.g. [9, Theorem 5, p. 94]).

Let (Z, Ω) be a symplectic manifold as in Section 4.1. A *symplectic connection* is a torsionfree C^∞ connection on Z that preserves the symplectic form Ω. A connection ∇ preserves the symplectic form if and only if for any two locally defined vector fields s and t on Z we have

$$\mathrm{d}\Omega(s, t) = \Omega(\nabla(s), t) + \Omega(s, \nabla(t)).$$

Given a symplectic connection, in [7], Fedosov proves the existence of a canonical quantization of Ω.

Let M be a complex manifold equipped with a G-structure $\mathfrak{p}$. It is easy to see that the symplectic manifold $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ in Theorem 4.2 has a canonical flat symplectic connection. To see this, given a vector space V, consider the unique connection on it preserved by translations and linear automorphisms. This connection is torsionfree and flat. Given a symplectic form ω on V, this connection is a symplectic connection, as it preserves the symplectic form. This connection clearly induces a symplectic connection on the symplectic manifold $(V', 2\omega')$ in Section 4.3 which is invariant under the action of $G := \mathrm{Sp}(V)/\mathbb{Z}_2$ on V'. Finally, since the symplectic manifold $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ is built from $(V', 2\omega')$ with transition functions in G, this symplectic connection on $(V', 2\omega')$ induces a symplectic connection on $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$. The symplectic connection on $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ is obviously flat.

If we consider the proof in [7] of the existence of a canonical quantization for a symplectic connection, then we first observe that the section r in [7, (3.3), p. 219] is identically zero for a flat symplectic connection. Indeed, this follows immediately from [7, Theorem 3.2]. From this observation it can be deduced that the explicit quantization of $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ constructed in Theorem 4.2 actually coincides with the canonical one obtained in [7]. In other words, Theorem 4.2 can be interpreted as an explicit realization, in the special situation under consideration, of the more abstract Fedosov quantization.

Given any quantization of a symplectic manifold there are some associated cohomological invariants [6,7,10]. Given a quantization of a symplectic manifold (Z, Ω), the corresponding cohomological invariant takes value in the direct sum

$$\frac{1}{\sqrt{-1}h} H^2(Z, \mathbb{C}) \oplus H^2(Z, \mathbb{C}[[h]]), \tag{4.6}$$

where h, as before, is a formal parameter (see p. 348 of [10]). The component in $(1/\sqrt{-1}h)H^2(Z, \mathbb{C})$ coincides with $(1/\sqrt{-1}h)$-times the de Rham cohomology class represented by the closed form Ω.

Let

$$\zeta(\mathfrak{p}) \in \frac{1}{\sqrt{-1}h} H^2(N'(\mathfrak{p}), \mathbb{C}) \oplus H^2(N'(\mathfrak{p}), \mathbb{C}[[h]])$$

be the cohomological invariant for the quantization in Theorem 4.2 for the G-structure $\mathfrak{p}$. Since in our situation the section r in [7, (3.3), p. 219] is identically 0, from [7, (3.2), p. 219] we know that the Weyl curvature is $(1/\sqrt{-1}h)$-times the symplectic form $\mathrm{d}\theta(\mathfrak{p})$. From this it follows immediately that the component of $\zeta(\mathfrak{p})$ in $H^2(N'(\mathfrak{p}), \mathbb{C}[[h]])$ vanishes (see the expression of the invariant in terms of Weyl curvature given in [10, p. 348]). Since the symplectic form $\mathrm{d}\theta(\mathfrak{p})$ on $N'(\mathfrak{p})$ is exact, we have $\zeta(\mathfrak{p}) = 0$.

If the cohomological invariants for two quantizations of a given symplectic manifold (Z, Ω) coincide, then there is an automorphism of $\mathcal{A}(Z)$, defined using differential operators, that takes one quantization to the other [10].

In the special case where M is a Riemann surface, we saw that the corresponding symplectic manifold $(N'(\mathfrak{p}), \mathrm{d}\theta(\mathfrak{p}))$ is independent of the projective structure $\mathfrak{p}$. Since the cohomological invariant vanish identically, we conclude that given two projective structures on a Riemann surface M, there is an automorphism of $\mathcal{A}(Z)$, where Z denote the space of all nonzero tangent vectors of M, that takes the quantization for one projective structure to the quantization for the other projective structure. In contrast, recall the earlier remark that a projective structure on a Riemann surface can be recovered from the corresponding quantization.

Given a G-structure on a complex manifold X of arbitrary dimension, there are some differential operators on X associated to it [3, Theorem 3.7, p. 10] which generalizes the construction of [2]. It is straight-forward to check that these differential operators can also be recovered directly from the quantization constructed in Theorem 4.2 for a G-structure.

Remark. There is another kind of quantization of a symplectic manifold known as geometric quantization [13]. For a symplectic manifold (Z, ω), consider the space of functions $C(Z)$ equipped with Poisson structure. A *geometric quantization* of (Z, ω) is defined by giving a Hilbert space $\mathcal{H}$ and a linear map

$$T_h : C(Z) \to A(\mathcal{H})$$

for every $h \in \mathbb{R}$, where $A(\mathcal{H})$ is the space of densely defined linear operators on $\mathcal{H}$, satisfying the condition

$$T_h(f) \circ T_h(g) - T_h(g) \circ T_h(f) = -\sqrt{-1} h T_h(\{f, g\})$$

for every $f, g \in C(Z)$.

If M is a complex manifold of dimension $2n + 1$ equipped with a contact structure, then consider the symplectic manifold $(N', \mathrm{d}\theta)$ in Proposition 2.1. Note that $(\mathrm{d}\theta)^{n+1} \wedge (\overline{\mathrm{d}\theta})^{n+1}$ defines a measure on N'. Let $\mathcal{H}$ be the space of square-integrable functions $L^2(N')$. For a function $f \in C(N')$, let $(\mathrm{d}\theta)^{-1}(\mathrm{d}f)$ denote the corresponding vector field. For any $h \in \mathbb{R}$ consider $T_h(f) \in A(L^2(N'))$ defined by

$$T_h(f)(\phi) = -\sqrt{-1} h X_f(\phi) - \theta(X_f)\phi + f\phi.$$

It is straight-forward to check that the pair $(L^2(N'), T)$ defines a geometric quantization of $(N', \mathrm{d}\theta)$.

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